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Lesson 14: Linear and Exponential Models—Comparing Growth Rates

Student Outcomes

* Students compare linear and exponential models by focusing on how the models change over intervals of equal length. Students observe from tables that a function that grows exponentially will eventually exceed a function that grows linearly.

Classwork

Example 1 (12 minutes)

Example 1

Linear Functions

* 1. Sketch points and . Are there values of and such that the graph of the linear function described by contains and ? If so, find those values. If not, explain why they do not exist.

The graph of the linear function contains when the -intercept, , is equal to . So all the linear functions whose graphs contain are represented by

, where is a real number.

For the graph of to also contain ,

The linear function whose graph passes through and is

.

* 1. Sketch and . Are there values of and so that the graph of a linear function described by contains and ? If so, find those values. If not, explain why they do not exist.

For a function , for each value of there can only be one value. Therefore, there is no linear function that contains and since each point has two different values for the same value.

Exponential Functions

Graphs (c) and (d) are both graphs of an exponential function of the form . Rewrite the function using the values of and required for the graph shown to be a graph of .



**Discussion (5 minutes)**

Discuss the rates of change in the Example 1.

* Consider Example 1a. Restrict the linear function to the positive integers, and consider the sequence …, i.e., the sequence . Ask students what they observe about this sequence, looking for the answer, “Each term is always the sum of the previous term and .”

Suggest they prove that statement by showing : for a positive integer, . Since , we see that .

In particular, the recursion formula shows that the linear function grows additively by over intervals of length (i.e., the length of the intervals between consecutive integers). This fact holds for any interval of length 1 (the equation holds for any real number , not just integers).

Similarly, for any linear function of the form , that linear function grows additively by over any interval of length 1. In fact, it grows additively by over any interval of length (if time permits, have students prove this by calculating !).

* Now consider Example 1c. Restrict the exponential function to the positive integers, and consider the sequence …, i.e., the sequence . Ask students what they observe about this sequence, looking for the answer, “Each term is always the product of the previous term and .”

Suggest they prove that statement by showing : for a positive integer, . Since , we see that .  
  
In particular, the recursion formula shows that the exponential function grows multiplicatively by 2 over intervals of length 1 (i.e., the length of the intervals between consecutive integers). This fact holds for any interval of length 1 (the equation holds for any real number , not just integers).  
  
Similarly, for any exponential function of the form , that exponential function grows multiplicatively by over any interval of length 1.

*Scaffolding:*

* For students performing above grade level: Ask students to make and test a conjecture about how an exponential function grows multiplicatively over any interval of length .  
    
  Answer:
* Moral: linear functions grow additively while exponential functions grow multiplicatively.

Example 2 (15 minutes)

Example 2

A lab researcher records the growth of the population of a yeast colony and finds that the population doubles every hour.

* 1. Complete the researcher’s table of data:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Hours into study |  |  |  |  |  |
| Yeast colony population (thousands) |  |  |  |  |  |

* 1. What is the exponential function that models the growth of the colony’s population?
  2. Several hours into the study, the researcher looks at the data and wishes there were more frequent measurements. Knowing that the colony doubles every hour, how can the researcher determine the population in half-hour increments? Explain.

Let represent the factor by which the population grows in half an hour. Since the population grows by the same factor in the next half hour, also , the population will grow by in hour. However the colony’s population doubles every hour:

The researcher should multiply the population by every half hour.

* 1. Complete the new table that includes half hour increments.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Hours into study |  |  |  |  |  |  |  |
| Yeast colony population (thousands) |  |  |  |  |  |  |  |

* 1. How would the calculation for the data change for time increments of minutes? Explain.

Now let represent the factor by which the population grows in minutes. Since the population grows by the same factor in the next minutes and the minutes after that, the population will grow by in hour. Since the colony’s population doubles every hour:

The researcher should multiply the population by every minutes.

* 1. Complete the new table that includes minute increments.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Hours into study |  |  |  |  |  |  |  |
| Yeast colony population (thousands) |  |  |  |  |  |  |  |

* 1. The researcher’s lab assistant studies the data recorded and makes the following claim:

Since the population doubles in hour, then half that growth happens in the first half hour and the second half of that growth happens in the second half hour. We should be able to find the population at by taking the average of the populations at and .

Is the assistant’s reasoning correct? Compare this strategy to your work in parts (c) and (e).

The assistant’s reasoning is not correct. By the assistant’s reasoning, the population growth at the first half hour mark would be because then the population would have grown by thousand cells from thousand cells in the first half hour and another thousand cells to thousand cells in the second half hour. This is linear growth, or the same amount of population growth n each half hour. However the percent growth by the assistant’s growth is from to and from to . For the population to double in hour, there must be constant percent growth in each half hour.

To the teacher: You might have a student who says that the assistant is correct if he is using the geometric mean. Recall that the geometric mean of two numbers is the square root of their product. In this case, it is , which is the same value as the half hour mark in part d. Explore with your students the connection between the answer in part (c) and definition of the geometric mean. Does the geometric mean give the same value as other half hour marks in the table to part (d)?

Example 3 (8 minutes)

Example 3

*Scaffolding:*

Encourage students to use graphing calculators to determine the linear and exponential regression functions. The steps for finding the exponential regression are listed below.

A California Population Projection Engineer in 1920 was tasked with finding a model that predicts the state’s population growth. He modeled the population growth as a function of time, years since . Census data shows that the population in , in thousands, was . In , the population of the state of California was thousand. He decided to explore both a linear and an exponential model.

* 1. Use the data provided to determine the equation of the linear function that models the population growth from to .
  2. Use the data provided and your calculator to determine the equation of the exponential function that models the population growth.
  3. Use the two functions to predict the population for the following years:

|  |  |  |  |
| --- | --- | --- | --- |
|  | Projected Population based on Linear Function,  (thousands) | Projected Population based on Exponential Function,  (thousands) | Census Population Data and Intercensal Estimates for California  (thousands) |
| 1935 |  |  |  |
| 1960 |  |  |  |
| 2010 |  |  |  |

* 1. Which function is a better model for the population growth of California in 1935 and in 1960?

The exponential model.

* 1. Does either model closely predict the population for 2010? What phenomenon explains the real population value?

Neither model closely predicts the population. After a population boom from 1900–1960, the population growth slows down. The following graph shows census and intercensal estimates for California’s population between 1900 and 2010.

Closing (2 minutes)

Lesson Summary

* **Given a linear function of the form and an exponential function of the form for a real number and constants and , consider the sequence given by and the sequence given by where . Both of these sequences can be written recursively:**

and , and   
 and .

**The first sequence shows that a linear function grows additively by the same summand over equal length intervals (i.e., the intervals between consecutive integers). The second sequence shows that an exponential function grows multiplicatively by the same factor over equal length intervals (i.e., the intervals between consecutive integers).**

* **An increasing exponential function will eventually exceed any linear function. That is, if is an exponential function with and , and is a linear function, then there is a real number such that for all , then . Sometimes this is not apparent in a graph displayed on a graphing calculator; that is because the graphing window does not show enough of the graphs for us to see the sharp rise of the exponential function in contrast with the linear function.**

Exit Ticket (3 minutes)

Name \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ Date\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

Lesson 14: Linear and Exponential Models—Comparing Growth Rates

Exit Ticket

A big company settles its new headquarters in a small city. The city council plans road construction based on traffic increasing at a linear rate, but based on the company’s massive expansion, traffic is really increasing exponentially.

What will be the repercussions of the city council’s current plans? Include what you know about linear and exponential growth in your discussion.

Exit Ticket Sample Solutions

A big company settles its new headquarters in a small city. The city council plans road construction based on traffic increasing at a linear rate, but based on the company’s massive expansion, traffic is really increasing exponentially.

What will be the repercussions of the city council’s current plans? Include what you know about linear and exponential growth in your discussion.

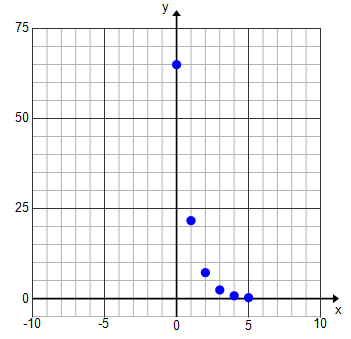
The city will not have the roads or capacity to handle the kind of traffic that will exist. Even if the linear growth of traffic initially outruns the exponential growth, eventually the exponential growth will catch up and exceed it. This means people will sit in traffic longer and the city will be generally congested for longer than if it had been planned for much heavier traffic flow.

Problem Set Sample Solutions

1. When a ball bounces up and down, the maximum height it reaches decreases with each bounce in a predictable way. Suppose for a particular type of squash ball dropped on a squash court, the maximum height, , after number of bounces can be represented by . How many times higher is the height after the first bounce compared to the height after the third bounce?

9 times higher

Graph the points for -values of .



1. Australia experienced a major pest problem in the early 20th century. The pest? Rabbits. In 1859, 24 rabbits were released by Thomas Austin at Barwon Park. In 1926, there were an estimated 10 billion rabbits in Australia. Needless to say, the Australian government spent a tremendous amount of time and money to get the rabbit problem under control. (To find more on this topic, visit Australia’s Department of Environment and Primary Industries website under Agriculture.)
   1. Based only on the information above, write an exponential function that would model Australia’s rabbit population growth.

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* 1. The model you created from the data in the problem is obviously a huge simplification from the actual function of the number of rabbits in any given year from 1859 to 1926. Name at least one complicating factor (about rabbits) that might make the graph of your function look quite different than the graph of the actual function.

Ex. A drought could have wiped out a huge percentage of rabbits in a single year, showing a dip in the graph of the actual function.

1. After graduating from college, Jane has two job offers to consider. Job A is compensated at a year but with no hope of ever having an increase in pay. Jane knows a few of her peers are getting that kind of an offer right out of college. Job B is for a social media start-up, which guarantees a mere a year. The founder is sure the concept of the company will be the next big thing in social networking and promises a pay increase of at the beginning of each new year.
   1. Which job will have a greater annual salary at the beginning of the 5th year? By approximately how much?

Job A, vs., a difference of about

* 1. Which job will have a greater annual salary at the beginning of the 10th year? By approximately how much?

Job A, vs., a difference of about

* 1. Which job will have a greater annual salary at the beginning of the 20th year? By approximately how much?

Job B, vs., a difference of about .

* 1. If you were in Jane’s shoes, which job would you take?

Answers will vary. Encourage students to voice reasons for each position. Note: The accumulated total after years for Job B is , and the accumulated total for Job A is . After 16.9 years, Job B begins to have a bigger total financial payoff than Job A.

1. The population of a town in 2007 is people. The town has gotten its fresh water supply from a nearby lake and river system with the capacity to provide water for up to people. Due to its proximity to a big city and a freeway, the town’s population has begun to grow more quickly than in the past. The table below shows the population counts for each year from 2007 to 2012.
   1. Write a function of that closely matches these data points for -values of .

|  |  |  |
| --- | --- | --- |
| Year | Years past 2007 | Population of the town |
| 2007 | 0 | 15,000 |
| 2008 | 1 | 15,600 |
| 2009 | 2 | 16,224 |
| 2010 | 3 | 16,873 |
| 2011 | 4 | 17,548 |
| 2012 | 5 | 18,250 |

The value of the ratio of population in one year to the population in the previous year appears to be the same for any two consecutive years in the table. The value of the ratio is , so a function that would model this data for -values of , is

* 1. Assume the function is a good model for the population growth from 2012 to 2032. At what year during the time frame 2012 to 2032 will the water supply be inadequate for the population?

If this model continues to hold true, the population will be larger than when is , which corresponds to the year 2025.